MIDTERM: VECTOR BUNDLES AND CHARACTERISTIC CLASSES

Date: **15th September 2024**
The Total points is **110** and the maximum you can score is **100** points.

(1) (5+5+5+5+5=25 points) Let $M \subset \mathbb{R}^A$ be a smooth manifold of dimension n.
   (a) Define the tangent space DM_x for a point $x \in M$ and the tangent manifold DM.
   (b) Define a smooth vector bundle on M.
   (c) Define tangent bundle on M.
   (d) When is M called parallelizable.
   (e) Show that the unit circle S^1 is parallelizable.

(2) (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth function. Using the definition, show that the graph of f, Γ with subspace topology, is a smooth manifold.

(3) (5+15=20 points) Let η be a vector bundle over a topological space B. Define Euclidean metric on η. Show that η and the dual vector bundle $\mathrm{Hom}(\eta, \epsilon^1)$ are isomorphic.

(4) (4+8+8=20 points) Define vector field of a smooth manifold. Show that if n is odd then the unit sphere S^n admits a vector field which is nowhere zero. Show that the tangent bundle of $\mathbb{P}^n$ has a trivial subbundle of rank 1.

(5) (5+15=20 points) Define total Stiefel-Whitney class of a vector bundle. Show that if n is even then tangent bundle of $\mathbb{P}^n$ does not have a subbundle of rank 1.

(6) (15 points) Let $G_n(\mathbb{R}^{n+k})$ be the set of n dimensional subspaces of $\mathbb{R}^{n+k}$. Show that $G_n(\mathbb{R}^{n+k})$ is a topological manifold of dimension nk.